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MAC-CPTM Situations Project

Situation ?? Ordering on complex numbers

Prompt

Is there an 'order' on the complex numbers like the inequality ordering ○ on the real numbers? Such an order would presumably allow one to compare any two complex number ***Z1*** and ***Z2*** with an inequality of the form ***Z1*** ○ ***Z2*** or ***Z2*** ○ ***Z1***. This question arose in two different mathematics content focused mathematics education courses for pre service secondary mathematics teachers.

In one class in Spring 2001 titled *Computer Graphics and Other Topics for Teaching Secondary School Geometry Concepts*, this topic arose and one student was asked to prepare and then give a presentation on complex numbers and orderings. In a second class in Fall 2001 titled *Foundations of Secondary Mathematics I* a question was posed for the class of how to put into order from smallest to largest, a group of numbers (including complex numbers) that had been left on a classroom blackboard.

Commentary

The foci are defining orderings on *n*-tuples of (real) numbers ☼^{n} and investigating the relations between chosen orderings and any arithmetic properties of the *n*-tuples of numbers involved. Key ideas are to relate the ordering choice on ☼^{n} with the usual ordering of real numbers ☼ , and the lexicographical ordering. Key ideas related to the compatibility of the chosen ordering and arithmetic properties are to see to what extent the chosen ordering is preserved under multiplications available in the cases where $n = 1$ (real numbers) or 2 (identified with complex numbers).

Mathematical Foci

Mathematical Focus 1

There are a variety of orderings ones can use for a total ordering of complex numbers and more generally n-tuples of real numbers. A total ordering ≤ on a set is a binary relation on the set which is transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$ for any elements x, y and z in the set), antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$), and total ($x \leq y$ or $y \leq x$). In a partial ordering not all elements are comparable. The lexicographical ordering is probably the most common ordering choice, and the difference between it and the usual alphabetical ordering in the dictionary (which orders n-tuples of letters where n can vary) may make for an interesting student exploration.

One natural attempt to compare complex numbers is to think about comparing the real and imaginary parts. By considering the 'canonical' bijection of $\mathbb{C} = \{ x + i^*y \mid x , y \in \mathbb{R}\}$ with $\mathbb{R}^2 = \{(x , y) \mid x , y \in \mathbb{R}\}$ given by $x + i^*y \to (x , y)$ we can define

$$\boldsymbol{Z1} = x_1 + i\, y_1 \leq \boldsymbol{Z2} = x_2 + i\, y_2$$

to mean either $x_1 < y_1$ (in which case we would write $\boldsymbol{Z1} < \boldsymbol{Z2}$)
or otherwise $x_1 = y_1$ and $x_2 \leq y_2$ (if $x_2 < y_2$ we would write $\boldsymbol{Z1} < \boldsymbol{Z2}$). This was in fact the ordering 'discovered' by a pre service secondary math teachers in one of the two classes mentioned above. His description was given geometrically in terms of the identification of $\mathbb{C}$ with $\mathbb{R}^2$ viewed as a two-dimensional plane in the usual way. To paraphrase, his description went as follows: if ***Z1*** is to the left of ***Z2***, then $\boldsymbol{Z1} < \boldsymbol{Z2}$. If ***Z1*** and ***Z2*** line up vertically, then the higher one is larger.

The lexicographical ordering is a method of imposing a total ordering on n-tuples of real numbers. If $\boldsymbol{X} = (x_1 , x_2 ,..., x_n), \boldsymbol{Y} = (y_1 , y_2 , ..., y_n) \in \mathbb{R}^n$ then we define $\boldsymbol{X} \leq \boldsymbol{Y}$ to mean either $x_1 < y_1$ or otherwise there is some $f \in \{ 1 , 2 ,..., n\text{-}1 \}$ such that $x_k = y_k$, for $k = 1 , 2,..., f$ and then $x_{f+1} < y_{f+1}$.

This is like the alphabetical order used in dictionaries with the 26-letter English alphabet applied to n-tuples of those letters (i.e. English words) where n, the number of letters in the words, is allowed to vary. The corresponding alphabetical ordering < can be interpreted to mean "occurs on a page with a smaller page number, or on the same page but closer to the top of the page, (and if there are several columns on each page, then the closer to the left side of the page the column is, the smaller (<) it counts compared to columns to its right) ".

Mathematical digression on alphabetical order:
There are interesting mathematical choices that have to be made to adapt the lexicographical ordering to words. One has to decide how to compare words with different numbers of letters. For example, the word *cat* would appear before the word *catapult*, but both appear before the word *do*.

In this case the lexicographical ordering is adapted to allow for comparisons of n-tuples and m-tulpes, $n , m \in \mathbb{N}$. If $\boldsymbol{X} = (x_1 , x_2 ,..., x_n) \in \mathbb{R}^n$ and $\boldsymbol{Y} = (y_1 , y_2 , ..., y_m) \in \mathbb{R}^m$ then we define $\boldsymbol{X} < \boldsymbol{Y}$ to mean $\boldsymbol{Y}$ to mean either one of the three mutually exclusive and exhaustive possibilities:

i. $x_1 < y_1$
ii. $n = min(n , m)$ and $x_k = y_k$, for $k = 1 , 2,..., n$
iii. there is some $f \in \{ 1 , 2 ,..., min(n , m) \text{-}1 \}$ such that $x_k = y_k$, for $k = 1 , 2,..., f$ and then $x_k < y_k$ for all $k \in \{ f\text{+}1,..., min(n , m) \}$.

In some references one finds the definition of alphabetical ordering in the dictionary defined by assuming all words have the same length. This is accomplished by adding 'blank' spaces to the end of all words. The common length of all the words so formed can presumably be taken to be a 'large' number, e.g. 100 letters, because one expects virtually all words will have fewer letters.

Giving a careful mathematical clarification of the relation between the usual definition of lexicographical ordering and that of dictionary alphabetical ordering might make a good mathematical and logical exercise for students. Their task might be to start with the lexicographical order definition on $\mathbb{R}^n$ and say precisely how it needs to be modified to explain the alphabetical ordering used in dictionaries.

The lexicographical ordering can be interpreted to be 'compatible' with the usual ordering on $\mathbb{R} = \mathbb{R}^1$ as follows. If we identify $\mathbb{R} = \{x \mid x \in \mathbb{R}\}$ with the set $R := \{(x, 0, 0, \ldots 0) \in \mathbb{R}^n \mid x \in \mathbb{R}\} \subset \mathbb{R}^n$ (the "x_1 axis") by the bijection $x \mapsto (x, 0, 0, \ldots 0) \in \mathbb{R}^n$ then the lexicographical ordering R inherits from $\mathbb{R}^n$ agrees with the usual ordering on $\mathbb{R}$ via the given bijection. Many other order-compatible copies of $\mathbb{R}$ in $\mathbb{R}^n$ can be described.
There is a subtle mathematical point in applying this idea to the case of $\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$. Any real number x is also a complex number, by considering x to be the same as the complex number as $x + i*0$. *If we try to relate inequalities of complex numbers (ordered via the lexicographical ordering) with inequalities of real numbers, we should be clear about the fact that it is legitimate to use the ordinary real number notion of inequality, even though comparisons of complex numbers are defined via the lexicographical ordering on* $\mathbb{R}^2$. For example, 2 < 4 is a correct inequality using the ordinary ordering on of $\mathbb{R}$ and also using the lexicographical ordering on $\mathbb{C}$. Similarly, 2 < 2 + i, but for this inequality to be understood the lexicographical ordering is the only one that can be applied sensibly.

Mathematical Focus 2
The ordering on the real numbers has a variety of properties which are important for solving and working with inequalities. An example includes this one: if $x, y \in \mathbb{R}$
$x, y \geq 0$ then $x * y \geq 0$. *One of the mathematical issues involved in the prompt is to determine to what extent any proposed ordering on complex numbers would or would not satisfy such properties.*

For the lexicographical ordering described above applied to $\mathbb{C} = \mathbb{R}^2$, we find for example, 0 < i and 0 < 2 + i. For ordinary real numbers, the product of two positives is a positive. But with the lexicographical ordering on $\mathbb{C}$ we find i *(2 + i) = -1 + 2*i < 0. So the lexicographical ordering on $\mathbb{C}$ does not have the same compatibility with multiplication that the usual ordering has on $\mathbb{R}$.

One can give other examples of pairs of complex numbers which are both positive or both negative, or one positive and the other negative, whose product is positive or negative. Below are additional examples showing complex numbers of any sign can be achieved with factors of any sign.

General Product Desired	**Example**
Positive*positive = positive	i*(2 – i) = 1 + 2*i >0
Positive*positive = negative	i *(2 + i) = -1 + 2*i < 0
Positive* negative = positive	i *(-2 - i) = 1 - 2*i > 0
Positive*negative = negative	i *(-2 + i) = -1 - 2*i < 0
negative*negative = positive	-i*(-2 + i) = 1 + 2* i > 0
negative*negative = negative	-i*(-2 - i) = -1 + 2* i < 0

Mathematical Focus 3
The lexicographical ordering on $\mathbb{C}$ failed to have the type of compatibility with

multiplication that the usual ordering on $\mathbb{R}$ (if $x, y \in \mathbb{R}$, $x, y > 0$ then $x * y > 0$). But might there be a different ordering on $\mathbb{C}$ which does have this type of compatibility?

Suppose we have a total ordering >' for $\mathbb{C}$ which is compatible with the usual ordering on $\mathbb{R}$. Assuming >' has a very reasonable-seeming property like either of the ones below:

if $x, y, z \in \mathbb{C}$ and $x >' y$ then $x + z >' y + z$ (**)

or

if $x >' 0$ then $0 >' -x$ (***)

Then this total ordering >' for $\mathbb{C}$, *cannot* have the desired compatibility with complex number multiplication.

To see this note, because >' is a total ordering, that we must have either i >' 0 or 0 >' i. Because >' is compatible with the usual ordering on $\mathbb{R}$, we have 0 >' -1 = i*i. If i >' 0, the inequality 0 >' -1 = i*i shows >' is not compatible with complex multiplication. If instead 0 >' i, then (**) or (***) implies -i >' 0 and then 0 >' -1 = (-i)*(-i) again shows >' is not compatible with complex multiplication.